Computational and Applied Mathematics (2021) 40:292
https://doi.org/10.1007/s40314-021-01685-2

An inverse time-fractional diffusion problem with Robin boundary condition in two layers spherical domain

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Received: 8 May 2021 / Revised: 3 October 2021 / Accepted: 17 October 2021

Abstract
In this paper, I am interested in the study of the inverse Cauchy boundary value problem for the time-fractional diffusion equation in two layers spherical domain. Given the data in the first layer, my goal is to recover the temperature distribution and the heat flux in the second layer. First, I prove that the problem is severely ill-posed in the Hadamard sense. After, I propose a truncation-type regularization approach enabling us to achieve Hölder-type error in L^2-norm between the regularized and exact solutions.

Keywords Time-fractional diffusion equation · Composite material · Fourier truncated method · Ill-posed problem · Robin boundary condition

Mathematics Subject Classification 26A33 · 30A10 · 35R25 · 47J06

1 Introduction

The inverse heat conduction problem (IHCP) has a very long-life history and has investigated by numerous researchers in recent decades. The problem comes from the practical situation that it is sometimes necessary to determine the temperature or heat flux on the surface of a body from a measured temperature at some fixed locations inside the body. Thus in this problem, one needs to recover the temperature and heat flux on an inaccessible portion of the body from the measurements on an accessible portion of the body. Theoretically speaking, it is well-known that the IHCP is an ill-posed problem in the sense of Hadamard. Unlike well-posed problems, standard numerical methods applying to ill-posed problems may fail to truly describe its solution due to its instability. As a result, there are many interesting regularization methods was proposed to deal with this kind of problem. For the IHCP in the single-layer material (S-IHCP), the literature is very abundant. For example: Carasso (1982)

Communicated by Vasily E. Tarasov.

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Published online: 03 November 2021

applied the well-known Tikhonov method to regularize the IHCP. Later on, Eldén (1987a, b) used the modified equation method to obtain a stable Hölder approximation to solution of IHCP. Other regularization methods which were designed for the S-IHCP can be found in Fu and Qiu (2003), Fu et al. (2005), Qiu et al. (2003), Seidman and Eldén (1990), Regińska (2001), Regińska and Eldén (1997), Cheng and Zhao (2020), Qian et al. (2007) and Ranjbar and Eldén (2014) which includes the optimal filtering method (Seidman and Eldén 1990), Wavelet–Galerkin method (Regińska 2001; Regińska and Eldén 1997), wavelet method (Fu and Qiu 2003; Qiu et al. 2003), modified method (Cheng and Ma 2017; Qian et al. 2007) and Fourier method (Fu et al. 2005; Khieu et al. 2019).

A composite material is a material consists of two or more separate materials with significantly different physical or chemical properties that, when combined, produce a material with characteristics different from the individual components. It is well-known that the composite materials have been used for a wide range of industrial applications nowadays, including piping, pressure vessels, fluid reservoirs, aerospace components, and naval structures (see Pavlou 2013; Kayhani et al. 2012 and the references therein). Recently, the heat conduction in composite material has attracted many researchers. For example: Berger and Karageorghis (1999) investigated the application of the method of fundamental solutions (MFS) to two-dimensional problems of steady-state heat conduction in composite material with linear boundary conditions. Later on, Karageorghis and Lesnic (2008) extended the work in Berger and Karageorghis (1999) to the case of nonlinear boundary conditions. In Jain et al. (2010), Jain et al. proposed an analytical study on the solution of the transient boundary-value problem of multi-layer heat conduction in the spherical coordinates. Niu et al. (2014) studied the inverse problem of determining the inner boundary location of heat conduction composite walls from the measurement data of temperature and heat flux on the exterior boundary. The problem is ill-posed and hence, the author applied well-known Tikhonov regularization technique to obtain a stable and accurate numerical approximation of the moving boundary. Very recently, Xiong et al. (2016) investigated the spatial inverse problem for a radially symmetric inverse heat conduction equation in a two-layer domain. The author considered a two-layer sphere domain, where the data is given in the first layer and from this data, the goal is to reconstruct the temperature distribution in the second layer. This problem is ill-posed in the sense of Hadamard and hence, they applied the classical Tikhonov regularization method to overcome the ill-posedness. It is noted that the boundary condition of the investigated problem in Xiong et al. (2016) is linear boundary condition of Neumann type.

While there are many interesting results on the direct inverse problems in the heat conduction in composite material with classical derivative, the literature on its fractional counterpart is still very limited. To the best of our knowledge, we didn't find any study concerning on the inverse heat conduction problem for fractional diffusion in composite material (FC-IHCP), even in the case of linear boundary condition (Dirichlet or Neumann boundary condition). Motivated by this and inspired by the above mentioned work, in the current paper, I am interested in investigating the fractional inverse Cauchy problem for the heat conduction in the composite sphere with nonlinear boundary condition of Robin type, i.e., the Dirichlet and Neumann boundary condition specify the temperature and the heat flow on the boundary, respectively. However, if the material is immersed in a surrounding medium held at some temperature, say T_0, then Newton's law of cooling states that the heat flow through the boundary of the domain is proportional to the temperature difference between the two mediums. This results into a Robin boundary condition. Specifically, let us consider a two-layer sphere that consist of the first layer in $0 \le r \le r_1$ and the second layer in $r_1 \le r \le R$. The two layers are in perfect thermal contact at $r = r_1$. Let $\eta_1, \eta_2 \in \mathbb{R}$, $k > 0$ be the thermal conductivity and $\alpha_1, \alpha_2 > 0$ be the thermal diffusivities of the first and the second layer, respectively. The temperature distributions in the first and the second layer, denoted by u_1 and u_2, respectively, satisfy the following conditions in the two domains $\mathbb{D}_1 := \{r | 0 < r \le r_1\}$ and $\mathbb{D}_2 := \{r | r_1 \le r \le R\}$:

- In the first layer $\mathbb{D}_1$:

$$\begin{cases} D_t^{\gamma_1} u_1(r,t) = \alpha_1 \left(\frac{2}{r} \frac{\partial u_1}{\partial r}(r,t) + \frac{\partial^2 u_1}{\partial r^2}(r,t) \right), & 0 < r < r_1, t > 0, \\ u_1(0,t) = \varphi(t), & t > 0, \\ \frac{\partial u_1}{\partial r}(0,t) = 0, & t > 0. \end{cases} \tag{1}$$

- In the second layer $\mathbb{D}_2$:

$$\begin{cases} D_t^{\gamma_2} u_2(r,t) = \alpha_2 \left(\frac{2}{r} \frac{\partial u_2}{\partial r}(r,t) + \frac{\partial^2 u_2}{\partial r^2}(r,t) \right), & r_1 < r \le R, t > 0, \\ u_2(r_1,t) = u_1(r_1,t), & t > 0, \\ \eta_2 u_2(r_1,t) + k \frac{\partial u_2}{\partial r}(r_1,t) = \eta_1 u_1(r_1,t) + k \frac{\partial u_1}{\partial r}(r_1,t), & t > 0, \end{cases} \tag{2}$$

and subject to homogeneous the initial conditions:

$$u_1(r,0) = u_2(r,0), \quad 0 < r < R. \tag{3}$$

Here, $D_t^{\gamma_i}$ is the Caputo fractional of order γ_i $(0 < \gamma_i \le 1, i = 1, 2)$ which is defined by

$$D_t^{\gamma_i} u(r,t) = \begin{cases} \frac{1}{\Gamma(1-\gamma_i)} \int_0^t \frac{u_s(r,s)}{(t-s)^{\gamma_i}} \mathrm{d}s, & 0 < \gamma_i < 1, \\ u_t(r,t), & \gamma_i = 1, \end{cases} \tag{4}$$

and φ is the exact data which is supposed to belong to $L^2(0,\infty)$. Thus, it is natural to assume that for any fixed $r \in [0, R]$, the temperature distribution u_1, u_2 and their derivatives $\frac{\partial u_1}{\partial r}$, $\frac{\partial u_2}{\partial r}$ belong to $L^2(0,\infty)$. The inverse Cauchy problem which will be investigated in the current paper is the problem of determining the temperature distribution u_2 and the flux distribution $\frac{\partial u_2}{\partial r}$ in the second layer from the measured data φ given in the first layer and the insulated condition at the accessible boundary $r = 0$.

When $\gamma_1 = \gamma_2 = 1$ and $\eta_1 = \eta_2 = 0$, the problem reduces to the one investigated in Xiong et al. (2016). The authors used the modified version of the classical Tikhonov regularization method achieving Hölder order convergence with respect to the L^2-norm for the temperature distribution and heat flux. As mentioned in Liu and Yamamoto (2010), the Caputo fractional derivatives for $0 < \gamma_i < 1$ at time t uses all the information about classical derivative f' for $(0, t)$, which is called the memory effect of the fractional derivatives. As a result, the fractional-order model are usually more adequate than the integer-order model. Recently, researchers are very interested in the fractional calculus because of the more and more convincing applications of fractional calculus in the real world. Here, with the appearance of the fractional derivative in time and the nonlinear boundary condition, the investigated problem is much more difficulty because of the non-locality in the fractional derivative and the ill-posedness of the problem. Here, I apply the Fourier technique to overcome the non-locality of the fractional derivative and a truncation-type regularization method to deal with the ill-posedness of the problem based on a priori the temperature distribution.

The organization of the paper: In Sect. 2, I present some auxiliary results which will be needed to prove the main results in next sections. Section 4 is devoted to the determination and L^2-error analysis of the temperature distribution. The determination and L^2-error analysis of the heat flux distribution is presented in Sect. 5. Finally, I end up the paper with a concluding remark to summarize the achievements related to the investigated problem.

2 Some auxiliary results

Throughout this paper, I extend all the functions to the whole line $-\infty < t < \infty$ by zero extension if necessary. I also assume that the measure data φ contains error that gives φ^δ satisfying

$$\left\|\varphi^\delta - \varphi\right\| \le \delta, \tag{5}$$

where the constant $\delta \in (0, 1)$ represent a bound on the measurement error and $\|\cdot\|$ denotes the L^2-norm. Assume that there exists a positive constant E so that the following a-priori bound exists for the solution u_2 of the problem (2):

$$\|u_2(R,\cdot)\|_{H^p(\mathbb{R})} := \left(\int_{\mathbb{R}} \left(1+\xi^2\right)^p |\widehat{u_2}(R,\xi)|^2 \mathrm{d}\xi\right)^{\frac{1}{2}} \le E, \tag{6}$$

where $p \ge 0$. Under the variable transformations $v(r,t) = ru_1(r,t)$ the system (1) becomes

$$\begin{cases} D_t^{\gamma_1} v(r,t) = \alpha_1 \frac{\partial^2 v}{\partial r^2}(r,t), & 0 < r < r_1, t > 0, \\ v(0,t) = 0, & t > 0, \\ \frac{\partial v}{\partial r}(0,t) = \varphi(t), & t > 0. \end{cases} \tag{7}$$

Let

$$\widehat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t)\, \mathrm{e}^{-i\xi t} \mathrm{d}t$$

be the Fourier transform of the function $g \in L^2(\mathbb{R})$. By applying Fourier transform to both sides of (7) with respect to t, one has

$$\begin{cases} (i\xi)^{\gamma_1} \widehat{v}(r,t) = \alpha_1 \widehat{v}_{rr}(r,t), & 0 < r < r_1, \\ \widehat{v}(0,\xi) = 0, \\ \widehat{v}_r(0,\xi) = \widehat{\varphi}(\xi), \end{cases} \tag{8}$$

where ξ is the variable of Fourier transform on t. Then the solution $\widehat{v}$ of problem (8) in the frequency domain for $0 < r \le r_1$:

$$\widehat{v}(r,\xi) = \sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}} r\right) \sqrt{\frac{\alpha_1}{(i\xi)^{\gamma_1}}} \widehat{\varphi}(\xi), \tag{9}$$

where

$$\sqrt{(i\xi)^{\gamma_1}} = \left|\frac{\xi}{2}\right|^{\frac{\gamma_1}{2}} (1 + i\,\mathrm{sign}(\xi))^{\gamma_1},$$

and $\mathrm{sign}(\xi)$ is the signum function of ξ. The solution v be recovered from using the inverse Fourier transform on $\widehat{v}$. From (9), it deduces that

$$\widehat{v}_r(r,\xi) = \cosh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}} r\right) \widehat{\varphi}(\xi).$$

Similarly, under the transformation $w(r,t) = ru_2(r,t)$ the system (2) becomes

$$\begin{cases} D_t^{\gamma_2} w(r,t) = \alpha_2 \frac{\partial^2 w}{\partial r^2}(r,t), & 0 < r < r_1, t > 0, \\ w(r_1,t) = v(r_1,t), & t > 0, \\ \eta_2 w(r_1,t) + k\frac{\partial w}{\partial r}(r_1,t) = \eta_1 v(r_1,t) + k\frac{\partial v}{\partial r}(r_1,t), & t > 0. \end{cases} \tag{10}$$

Applying Fourier transform to (10) with respect to t, we have

$$\begin{cases} (i\xi)^{\gamma_2}\widehat{w}(r,t) = \alpha_2 \widehat{w}_{rr}(r,t), & r_1 < r < R, \\ \widehat{w}(r_1,\xi) = \widehat{v}(r_1,\xi), \\ \eta_2\widehat{w}(r_1,\xi) + k\widehat{w}_r(r_1,\xi) = \eta_1\widehat{v}(r_1,\xi) + k\widehat{v}_r(r_1,\xi). \end{cases} \tag{11}$$

We already had $\widehat{u}_2(r,\xi) = \frac{1}{r}\widehat{w}(r,\xi)$ and $\frac{\partial\widehat{u}_2(r,\xi)}{\partial r} = \frac{1}{r}\frac{\partial\widehat{w}(r,\xi)}{\partial r} - \frac{1}{r^2}\widehat{w}(r,\xi)$. It follows that

$$\widehat{u}_2(r,\xi) = \widehat{A}(r,\xi)\widehat{\varphi}(\xi) = \left(\widehat{A}_1(r,\xi) + \widehat{A}_2(r,\xi) + \widehat{A}_3(r,\xi)\right)\widehat{\varphi}(\xi), \tag{12}$$

$$\frac{\partial\widehat{u}_2(r,\xi)}{\partial r} = \widehat{X}(r,\xi)\widehat{\varphi}(\xi) = \left(\widehat{X}_1(r,\xi) + \widehat{X}_2(r,\xi) + \widehat{X}_3(r,\xi)\right)\widehat{\varphi}(\xi), \tag{13}$$

where

$$\widehat{A}_1(r,\xi) = \frac{\sqrt{\alpha_1\alpha_2}(\eta_1-\eta_2)}{rk\sqrt{(i\xi)^{\gamma_1+\gamma_2}}}\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right),$$

$$\widehat{A}_2(r,\xi) = \frac{\sqrt{\alpha_1}}{r\sqrt{(i\xi)^{\gamma_1}}}\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right),$$

$$\widehat{A}_3(r,\xi) = \frac{\sqrt{\alpha_2}}{r\sqrt{(i\xi)^{\gamma_2}}}\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right),$$

$$\widehat{X}_1(r,\xi) = \frac{\sqrt{\alpha_1}(\eta_1-\eta_2)}{rk\sqrt{(i\xi)^{\gamma_1}}}\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right),$$

$$\widehat{X}_2(r,\xi) = \frac{\sqrt{\alpha_1}}{r\sqrt{\alpha_2}\sqrt{(i\xi)^{\gamma_1-\gamma_2}}}\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right),$$

$$\widehat{X}_3(r,\xi) = \frac{1}{r}\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right) - \frac{1}{r}\widehat{A}(r,\xi).$$

The solution u_2 and heat flux $\frac{\partial u_2}{\partial r}$ can be recovered from taking the inverse Fourier transforms on (12) and (13). Next, we have the following auxiliary results.

Lemma 2.1 (Xiong et al. 2016) *For $x > 0$ and $\eta \in \mathbb{C}$. Then we have the following estimates.*

1. $|\sinh(\eta x)|, |\cosh(\eta x)| \le e^{|\eta|x}$,
2. $\left|\frac{\sinh(\eta x)}{\eta}\right| \le x e^{|\eta|x}$,
3. $|\sinh\eta|, |\cosh\eta| \ge \sinh|\Re(\eta)|$.

Lemma 2.2 *For $r \in (r_1, R]$ and $\xi \in \mathbb{R}$. Then, we have the following estimates.*

1. $\left|\widehat{A}(r,\xi)\right|, \left|\widehat{X}(r,\xi)\right| \le c_1 e^{\Psi(r,\xi)}$,

2. $\left|\widehat{X}(r,\xi)\right| \le \overline{c}_1(\xi)\,\mathrm{e}^{\Psi(r,\xi)}$,

where

$$\Psi(r,\xi) = \frac{r_1}{\sqrt{\alpha_1}}|\xi|^{\frac{\gamma_1}{2}} + \frac{r-r_1}{\sqrt{\alpha_2}}|\xi|^{\frac{\gamma_2}{2}},$$
$$c_1 = 2 + \frac{r_1|\eta_1-\eta_2|}{k},$$
$$\overline{c}_1(\xi) = \frac{|\eta_1-\eta_2|}{k} + \frac{1+c_1}{r_1} + \frac{|\xi|^{\frac{\gamma_2}{2}}}{\sqrt{\alpha_2}}.$$

Proof (1) For $\widehat{A}(r,\xi)$. In view of Lemma 2.1, we obtain that

$$\begin{aligned}
\left|\widehat{A}(r,\xi)\right| &\le \frac{|\eta_1-\eta_2|}{rk}\left|\frac{\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)}{\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}}\right|\left|\frac{\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)}{\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}}\right| \\
&\quad + \frac{1}{r}\left|\frac{\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)}{\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}}\right|\left|\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)\right| \\
&\quad + \frac{1}{r}\left|\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\right|\left|\frac{\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)}{\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}}\right| \\
&\le \left(2 + \frac{r_1|\eta_1-\eta_2|}{k}\right)\mathrm{e}^{\frac{r_1}{\sqrt{\alpha_1}}|\xi|^{\frac{\gamma_1}{2}} + \frac{r-r_1}{\sqrt{\alpha_2}}|\xi|^{\frac{\gamma_2}{2}}} = c_1\mathrm{e}^{\Psi(r,\xi)}.
\end{aligned}$$

(2) For $\widehat{X}(r,\xi)$. By applying Lemma 2.1 again, we get that

$$\begin{aligned}
\left|\widehat{X}(r,\xi)\right| &\le \frac{|\eta_1-\eta_2|}{r_1 k}\left|\frac{\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)}{\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}}\right|\left|\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)\right| \\
&\quad + \frac{|\xi|^{\frac{\gamma_2}{2}}}{r_1\sqrt{\alpha_2}}\left|\frac{\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)}{\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}}\right|\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)\right| \\
&\quad + \frac{1}{r_1}\left|\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\right|\left|\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)\right| + \frac{1}{r_1}\left|\widehat{A}(r,\xi)\right| \\
&\le \left(\frac{|\eta_1-\eta_2|}{k} + \frac{1+c_1}{r_1} + \frac{|\xi|^{\frac{\gamma_2}{2}}}{\sqrt{\alpha_2}}\right)\mathrm{e}^{\frac{r_1}{\sqrt{\alpha_1}}|\xi|^{\frac{\gamma_1}{2}} + \frac{r-r_1}{\sqrt{\alpha_2}}|\xi|^{\frac{\gamma_2}{2}}} = \overline{c}_1(\xi)\,\mathrm{e}^{\Psi(r,\xi)}.
\end{aligned}$$

The proof is completed. □

Lemma 2.3 *For $r \in (r_1, R]$ and $|\xi| \geq 1$. Then we have the following estimates.*

1. $\left|\widehat{A}(r,\xi)\right| \leq \frac{c_2}{|\xi|^{\frac{\gamma^-}{2}}} e^{\Phi(r,\xi)}$,
2. $\left|\widehat{X}(r,\xi)\right| \leq c_3 |\xi|^{\frac{\gamma^*}{2}} e^{\Phi(r,\xi)}$,

where

$$\gamma^- = \min\{\gamma_1, \gamma_2\},$$
$$\gamma^* = \begin{cases} 0, & \text{if } \gamma_1 \geq \gamma_2, \\ \gamma_2 - \gamma_1, & \text{if } \gamma_1 < \gamma_2, \end{cases}$$
$$c_2 = \frac{2\sqrt{2}}{r_1}\left(\sqrt{\alpha_1} + \sqrt{\alpha_2} + \frac{\sqrt{\alpha_1\alpha_2}\,|\eta_1 - \eta_2|}{k}\right),$$
$$c_3 = \frac{2\sqrt{2}}{r_1}\left(1 + \frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}} + \frac{\sqrt{\alpha_1}\,|\eta_1 - \eta_2|}{k}\right) + \frac{c_2}{r_1},$$
$$\Phi(r,\xi) = \frac{r_1}{\sqrt{\alpha_1}}\cos\left(\frac{\pi}{4}\gamma_1\right)|\xi|^{\frac{\gamma_1}{2}} + \frac{r - r_1}{\sqrt{\alpha_2}}\cos\left(\frac{\pi}{4}\gamma_2\right)|\xi|^{\frac{\gamma_2}{2}}.$$

Proof (1) For $\widehat{A}(r,\xi)$. We denote

$$a := a(\xi) = \sqrt{\frac{|\xi|^{\gamma_1}}{\alpha_1}} r_1 \cos\left(\frac{\pi}{4}\gamma_1\right), \quad \overline{a} := \overline{a}(\xi) = \sqrt{\frac{|\xi|^{\gamma_1}}{\alpha_1}} r_1 \sin\left(\frac{\pi}{4}\gamma_1\right)\operatorname{sign}(\xi),$$
$$b(r) := b(r,\xi) = \sqrt{\frac{|\xi|^{\gamma_2}}{\alpha_2}}(r - r_1)\cos\left(\frac{\pi}{4}\gamma_2\right), \quad \overline{b}(r) := \overline{b}(r,\xi)$$
$$= \sqrt{\frac{|\xi|^{\gamma_2}}{\alpha_2}}(r - r_1)\sin\left(\frac{\pi}{4}\gamma_2\right)\operatorname{sign}(\xi).$$

It is easy to see that

$$\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}} r_1 = \sqrt{\frac{|\xi|^{\gamma_1}}{\alpha_1}} r_1\left(\cos\left(\frac{\pi}{4}\gamma_1\right) + i\sin\left(\frac{\pi}{4}\gamma_1\right)\operatorname{sign}(\xi)\right),$$
$$\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r - r_1) = \sqrt{\frac{|\xi|^{\gamma_2}}{\alpha_2}}(r - r_1)\left(\cos\left(\frac{\pi}{4}\gamma_2\right) + i\sin\left(\frac{\pi}{4}\gamma_2\right)\operatorname{sign}(\xi)\right).$$

Therefore

$$\begin{aligned}
\left|\widehat{A}(r,\xi)\right| \leq{}& \frac{\sqrt{\alpha_1\alpha_2}\,|\eta_1 - \eta_2|}{r_1 k|\xi|^{\frac{\gamma_1+\gamma_2}{2}}}\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}} r_1\right)\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r - r_1)\right)\right| \\
&+ \frac{\sqrt{\alpha_1}}{r_1|\xi|^{\frac{\gamma_1}{2}}}\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}} r_1\right)\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r - r_1)\right)\right| \\
&+ \frac{\sqrt{\alpha_2}}{r_1|\xi|^{\frac{\gamma_2}{2}}}\left|\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}} r_1\right)\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r - r_1)\right)\right| \\
\leq{}& \frac{\sqrt{\alpha_1\alpha_2}\,|\eta_1 - \eta_2|}{r_1 k|\xi|^{\frac{\gamma^-}{2}}}\left|\sinh(a + i\overline{a})\sinh\left(b(r) + i\overline{b}(r)\right)\right|
\end{aligned}$$

$$+\frac{\sqrt{\alpha_1}}{r_1|\xi|^{\frac{\gamma^-}{2}}}\left|\sinh\left(a+i\overline{a}\right)\cosh\left(b\left(r\right)+i\overline{b}\left(r\right)\right)\right|$$
$$+\frac{\sqrt{\alpha_2}}{r_1|\xi|^{\frac{\gamma^-}{2}}}\left|\cosh\left(a+i\overline{a}\right)\sinh\left(b\left(r\right)+i\overline{b}\left(r\right)\right)\right|.$$

By applying the following formulas

$$\sinh\left(x+iy\right)=\sinh x\cos y+i\cosh x\sin y\quad\text{for all } x,y\in\mathbb{R},$$
$$\cosh\left(x+iy\right)=\cosh x\cos y+i\sinh x\sin y\quad\text{for all } x,y\in\mathbb{R},$$

and Lemma 2.1, we conclude that

$$\left|\widehat{A}\left(r,\xi\right)\right|\leq\frac{2\sqrt{2}}{r_1|\xi|^{\frac{\gamma^-}{2}}}\left(\sqrt{\alpha_1}+\sqrt{\alpha_2}+\frac{\sqrt{\alpha_1\alpha_2}\left|\eta_1-\eta_2\right|}{k}\right)\mathrm{e}^{a+b(r)}=\frac{c_2}{|\xi|^{\frac{\gamma^-}{2}}}\mathrm{e}^{\Phi(r,\xi)}.$$

(2) For $\widehat{X}\left(r,\xi\right)$. By the same arguments as the above, we have

$$\begin{aligned}\left|\widehat{X}\left(r,\xi\right)\right|\leq&\frac{\sqrt{\alpha_1}\left|\eta_1-\eta_2\right|}{r_1k}\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}\left(r-r_1\right)\right)\right|\\&+\frac{\sqrt{\alpha_1}}{r_1\sqrt{\alpha_2}}|\xi|^{\frac{\gamma_2-\gamma_1}{2}}\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}\left(r-r_1\right)\right)\right|\\&+\frac{1}{r_1}\left|\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}\left(r-r_1\right)\right)\right|+\frac{c_2}{r_1}\mathrm{e}^{\Phi(r,\xi)}\\\leq&\left(\frac{2\sqrt{2}}{r_1}\left(1+\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}}+\frac{\sqrt{\alpha_1}\left|\eta_1-\eta_2\right|}{k}\right)+\frac{c_2}{r_1}\right)|\xi|^{\frac{\gamma^*}{2}}\mathrm{e}^{\Phi(r,\xi)}\\=&c_3|\xi|^{\frac{\gamma^*}{2}}\mathrm{e}^{\Phi(r,\xi)}.\end{aligned}$$

The proof is completed. □

Lemma 2.4 *For $r\in\left(r_1,R\right]$. Then we have*

$$\left|\widehat{A}\left(r,\xi\right)\right|\geq\frac{c_4}{|\xi|^{\frac{\gamma^+}{2}}}\mathrm{e}^{\Phi(r,\xi)}\quad\text{for all }\ |\xi|\geq\theta_{\max}\left(r\right),$$

in which

$$\gamma^+=\max\left\{\gamma_1,\gamma_2\right\},$$
$$c_4=\min\left\{\frac{\sqrt{\alpha_1}}{32R}\left(1+\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\right),\frac{\sqrt{\alpha_1}}{2R},\frac{\sqrt{\alpha_2}}{2R}\right\},$$
$$\theta_{\max}\left(r\right)=\begin{cases}\theta_1\left(r\right), & \text{if } \gamma_1=\gamma_2,\\\theta_2\left(r\right), & \text{if } \gamma_1>\gamma_2,\\\theta_3\left(r\right), & \text{if } \gamma_1<\gamma_2,\end{cases}$$
$$\theta_{\max}:=\theta_{\max}\left(R\right).$$

where

$$\theta_1(r)=\max\left\{\left(\frac{\sqrt{\alpha_1}\ln 2}{r_1\cos\left(\frac{\pi}{4}\gamma\right)}\right)^{\frac{2}{\gamma}},\left(\frac{\sqrt{\alpha_2}\ln 2}{(r-r_1)\cos\left(\frac{\pi}{4}\gamma\right)}\right)^{\frac{2}{\gamma}},\left(\frac{32\sqrt{\alpha_1\alpha_2}\,|\eta_1-\eta_2|}{k\left(\sqrt{\alpha_1}+\sqrt{\alpha_2}\right)}\right)^{\frac{2}{\gamma}}\right\},$$

$$\theta_2(r)=\max\left\{\omega_{\max},\left(\frac{32\sqrt{\alpha_1}}{\sqrt{\alpha_2}}\right)^{\frac{2}{\gamma_1-\gamma_2}},\left(\frac{2\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k}\right)^{\frac{2}{\gamma_2}}\right\},$$

$$\theta_3(r)=\max\left\{\omega_{\max},\left(\frac{32\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\right)^{\frac{2}{\gamma_2-\gamma_1}},\left(\frac{2\sqrt{\alpha_1}\,|\eta_1-\eta_2|}{k}\right)^{\frac{2}{\gamma_1}}\right\},$$

with

$$\omega_{\max}(r)=\max\left\{\left(\frac{\sqrt{\alpha_1}\ln 2}{2r_1\cos\left(\frac{\pi}{4}\gamma_1\right)}\right)^{\frac{2}{\gamma_1}},\left(\frac{\sqrt{\alpha_2}\ln 2}{2(r-r_1)\cos\left(\frac{\pi}{4}\gamma_2\right)}\right)^{\frac{2}{\gamma_2}}\right\},$$
$$\omega_{\max}:=\omega_{\max}(R).$$

Proof We denote

$$B_1(r):=\sinh(a+i\overline{a})\cosh\left(b(r)+i\overline{b}(r)\right),$$
$$B_2(r):=\cosh(a+i\overline{a})\sinh\left(b(r)+i\overline{b}(r)\right),$$
$$B_3(r):=\sinh(a+i\overline{a})\sinh\left(b(r)+i\overline{b}(r)\right).$$

It is not difficult to verify that

$$B_1(r)=\frac{1}{2}\left(\sinh\left(a+b(r)+i\left(\overline{a}+\overline{b}(r)\right)\right)+\sinh\left(a-b(r)+i\left(\overline{a}-\overline{b}(r)\right)\right)\right),$$
$$B_2(r)=\frac{1}{2}\left(\sinh\left(a+b(r)+i\left(\overline{a}+\overline{b}(r)\right)\right)+\sinh\left(-a+b(r)+i\left(-\overline{a}+\overline{b}(r)\right)\right)\right),$$
$$B_3(r)=\frac{1}{2}\left(\cosh\left(a+b(r)+i\left(\overline{a}+\overline{b}(r)\right)\right)-\cosh\left(a-b(r)+i\left(\overline{a}-\overline{b}(r)\right)\right)\right).$$

Then we have

$$\begin{aligned}\left|\widehat{A}(r,\xi)\right|\geq\frac{\sqrt{\alpha_1}}{R|\xi|^{\frac{\gamma^+}{2}}}&\left|\left|\sqrt{(i\xi)^{\gamma^+-\gamma_1}}\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)\right.\right.\\&+\left.\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\sqrt{(i\xi)^{\gamma^+-\gamma_2}}\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)\right|\\&-\left.\frac{\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k\sqrt{(i\xi)^{\gamma_1+\gamma_2-\gamma^+}}}\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)\right|\right|\\=\frac{\sqrt{\alpha_1}}{R|\xi|^{\frac{\gamma^+}{2}}}&\left|\left|\sqrt{(i\xi)^{\gamma^+-\gamma_1}}B_1(r)+\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\sqrt{(i\xi)^{\gamma^+-\gamma_2}}B_2(r)\right|-\frac{\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k\sqrt{(i\xi)^{\gamma_1+\gamma_2-\gamma^+}}}B_3(r)\right|.\end{aligned}$$

Between γ_1 and γ_2 there are three cases:

In case 1: $\gamma_1 = \gamma_2 =: \gamma$ it follows that

$$\begin{aligned}
\left|\widehat{A}(r,\xi)\right| &\geq \frac{\sqrt{\alpha_1}}{2R|\xi|^{\frac{\gamma}{2}}}\Bigg\{\Bigg|\left(1+\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\right)\sinh\left(a+b(r)+i\left(\overline{a}+\overline{b}(r)\right)\right)\\
&\quad+\sinh\left(a-b(r)+i\left(\overline{a}-\overline{b}(r)\right)\right)+\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\sinh\left(-a+b(r)+i\left(-\overline{a}+\overline{b}(r)\right)\right)\Bigg|\\
&\quad-\frac{\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k|\xi|^{\frac{\gamma}{2}}}\left(\left|\cosh\left(a+b(r)+i\left(\overline{a}+\overline{b}(r)\right)\right)\right|+\left|\cosh\left(a-b(r)+i\left(\overline{a}-\overline{b}(r)\right)\right)\right|\right)\Bigg\}\\
&\geq \frac{\sqrt{\alpha_1}}{2R|\xi|^{\frac{\gamma}{2}}}\Bigg\{\left(1+\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\right)\left|\sinh\left(a+b(r)+i\left(\overline{a}+\overline{b}(r)\right)\right)\right|\\
&\quad-\left|\sinh\left(a-b(r)+i\left(\overline{a}-\overline{b}(r)\right)\right)\right|-\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\left|\sinh\left(-a+b(r)+i\left(-\overline{a}+\overline{b}(r)\right)\right)\right|\\
&\quad-\frac{\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k|\xi|^{\frac{\gamma}{2}}}\left(\left|\cosh\left(a+b(r)+i\left(\overline{a}+\overline{b}(r)\right)\right)\right|+\left|\cosh\left(a-b(r)+i\left(\overline{a}-\overline{b}(r)\right)\right)\right|\right)\Bigg\}.
\end{aligned}$$

By applying Lemma 2.1, we obtain that

$$\begin{aligned}
\left|\widehat{A}(r,\xi)\right| \geq \frac{\sqrt{\alpha_1}}{4R|\xi|^{\frac{\gamma}{2}}}\Bigg\{&\left(1+\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\right)\left(\mathrm{e}^{a+b(r)}-\mathrm{e}^{a-b(r)}-\mathrm{e}^{-a+b(r)}-\mathrm{e}^{-a-b(r)}\right)\\
&-\frac{4\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k|\xi|^{\frac{\gamma}{2}}}\mathrm{e}^{a+b(r)}\Bigg\}.
\end{aligned}$$

Due to $|\xi| \geq \max\left\{\left(\frac{\sqrt{\alpha_1}\ln 2}{r_1\cos(\frac{\pi}{4}\gamma)}\right)^{\frac{2}{\gamma}}, \left(\frac{\sqrt{\alpha_2}\ln 2}{(r-r_1)\cos(\frac{\pi}{4}\gamma)}\right)^{\frac{2}{\gamma}}\right\}$, one has

$$\left|\widehat{A}(r,\xi)\right| \geq \frac{\sqrt{\alpha_1}}{2R|\xi|^{\frac{\gamma}{2}}}\left\{\frac{1}{8}\left(1+\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\right)-\frac{2\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k|\xi|^{\frac{\gamma}{2}}}\right\}\mathrm{e}^{\Phi(r,\xi)}.$$

On the other hand, since $|\xi| \geq \left(\frac{32\sqrt{\alpha_1\alpha_2}|\eta_1-\eta_2|}{k(\sqrt{\alpha_1}+\sqrt{\alpha_2})}\right)^{\frac{2}{\gamma}}$ it follows that

$$\left|\widehat{A}(r,\xi)\right| \geq \frac{\sqrt{\alpha_1}}{32R}\left(1+\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\right)\frac{1}{|\xi|^{\frac{\gamma}{2}}}\mathrm{e}^{\Phi(r,\xi)} \geq \frac{c_4}{|\xi|^{\frac{\gamma^+}{2}}}\mathrm{e}^{\Phi(r,\xi)}.$$

In case 2: $\gamma_1 > \gamma_2$. By the same arguments the above, we obtain the following sequential results

$$\begin{aligned}
\left|\widehat{A}(r,\xi)\right| &\geq \frac{\sqrt{\alpha_1}}{R|\xi|^{\frac{\gamma_1}{2}}}\left\{\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}|\xi|^{\frac{\gamma_1-\gamma_2}{2}}\left|\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\right|\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)\right|\right.\\
&\quad-\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\right|\left|\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)\right|\\
&\quad\left.-\frac{\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k|\xi|^{\frac{\gamma_2}{2}}}\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\right|\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)\right|\right\}\\
&\geq \frac{\sqrt{\alpha_1}}{R|\xi|^{\frac{\gamma_1}{2}}}\left(\frac{\sqrt{\alpha_2}}{4\sqrt{\alpha_1}}|\xi|^{\frac{\gamma_1-\gamma_2}{2}}\left(e^{a}-e^{-a}\right)\left(e^{b(r)}-e^{-b(r)}\right)-e^{a+b(r)}-\frac{\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k|\xi|^{\frac{\gamma_2}{2}}}e^{a+b(r)}\right)\\
&\geq \frac{\sqrt{\alpha_1}}{R|\xi|^{\frac{\gamma_1}{2}}}\left(\frac{\sqrt{\alpha_2}}{16\sqrt{\alpha_1}}|\xi|^{\frac{\gamma_1-\gamma_2}{2}}-1-\frac{\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k|\xi|^{\frac{\gamma_2}{2}}}\right)e^{\Phi(r,\xi)}\\
&\geq \frac{\sqrt{\alpha_1}}{R|\xi|^{\frac{\gamma_1}{2}}}\left(1-\frac{\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k|\xi|^{\frac{\gamma_2}{2}}}\right)e^{\Phi(r,\xi)}\geq\frac{\sqrt{\alpha_1}}{2R|\xi|^{\frac{\gamma_1}{2}}}e^{\Phi(r,\xi)}\geq\frac{c_4}{|\xi|^{\frac{\gamma^+}{2}}}e^{\Phi(r,\xi)}.
\end{aligned}$$

In case 3: $\gamma_1 < \gamma_2$. By the same the arguments the above, we get the following sequential results

$$\begin{aligned}
\left|\widehat{A}(r,\xi)\right| &\geq \frac{\sqrt{\alpha_1}}{R|\xi|^{\frac{\gamma_2}{2}}}\left\{|\xi|^{\frac{\gamma_2-\gamma_1}{2}}\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\right|\left|\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)\right|\right.\\
&\quad-\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\left|\cosh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\right|\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)\right|\\
&\quad\left.-\frac{\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k|\xi|^{\frac{\gamma_1}{2}}}\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_1}}{\alpha_1}}r_1\right)\right|\left|\sinh\left(\sqrt{\frac{(i\xi)^{\gamma_2}}{\alpha_2}}(r-r_1)\right)\right|\right\}\\
&\geq \frac{\sqrt{\alpha_1}}{R|\xi|^{\frac{\gamma_2}{2}}}\left(\frac{1}{4}|\xi|^{\frac{\gamma_2-\gamma_1}{2}}\left(e^{a}-e^{-a}\right)\left(e^{b(r)}-e^{-b(r)}\right)-\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}e^{a+b(r)}-\frac{\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k|\xi|^{\frac{\gamma_1}{2}}}e^{a+b(r)}\right)\\
&\geq \frac{\sqrt{\alpha_1}}{R|\xi|^{\frac{\gamma_2}{2}}}\left(\frac{1}{16}|\xi|^{\frac{\gamma_2-\gamma_1}{2}}-\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}-\frac{\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k|\xi|^{\frac{\gamma_1}{2}}}\right)e^{\Phi(r,\xi)}\\
&\geq \frac{\sqrt{\alpha_1}}{R|\xi|^{\frac{\gamma_2}{2}}}\left(\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}-\frac{\sqrt{\alpha_2}\,|\eta_1-\eta_2|}{k|\xi|^{\frac{\gamma_1}{2}}}\right)e^{\Phi(r,\xi)}\geq\frac{\sqrt{\alpha_2}}{2R|\xi|^{\frac{\gamma_2}{2}}}e^{\Phi(r,\xi)}\geq\frac{c_4}{|\xi|^{\frac{\gamma^+}{2}}}e^{\Phi(r,\xi)}.
\end{aligned}$$

The proof is completed. □

3 The ill-posedness

In this section, I prove the ill-posedness of the problem (1)–(2). More precisely, we have the following theorem.

Theorem 3.1 *The problem* (2) *is ill-posed in the Hadamard sense with respect to the* L^2*-norm.*

Proof The following example demonstrates the ill-posedness of (2). For any $n \in \mathbb{N}$ with $n \geq \theta_{\max}(r)$. We denote $\Omega_n := \{\xi \in \mathbb{R}; n \leq \xi \leq n+1\}$, let $\varphi_n \in L^2(\mathbb{R})$ be the measured data, such that

$$\widehat{\varphi_n}(\xi) = \begin{cases} \widehat{\varphi}(\xi) + \frac{1}{n}, & \text{if } \xi \in \Omega_n, \\ \widehat{\varphi}(\xi), & \text{if } \xi \in \mathbb{R}\backslash\Omega_n. \end{cases}$$

By applying Parseval's identity, it follows that

$$\|\varphi_n - \varphi\| = \|\widehat{\varphi_n} - \widehat{\varphi}\| = \left(\int_{\Omega_n} \frac{1}{n^2} \mathrm{d}(\xi) \right)^{\frac{1}{2}} = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

Let u_2 and u_{2n} be two solutions of (2) corresponding to the data φ and φ_n, respectively, that is

$$\widehat{u_2}(r,\xi) = \widehat{A}(r,\xi)\,\widehat{\varphi}(\xi) \quad \text{and} \quad \widehat{u_{2n}}(r,\xi) = \widehat{A}(r,\xi)\,\widehat{\varphi_n}(\xi).$$

By Parseval's identity and Lemma 2.4, it follows that

$$\begin{aligned} \|u_{2n}(r,\cdot) - u_2(r,\cdot)\|^2 &= \|\widehat{u}_{2n}(r,\cdot) - \widehat{u}_2(r,\cdot)\|^2 = \frac{1}{n^2} \int_n^{n+1} \left|\widehat{A}(r,\xi)\right|^2 \mathrm{d}\xi \\ &\geq \frac{c_4^2}{n^2} \int_n^{n+1} \frac{\mathrm{e}^{2\Phi(r,\xi)}}{|\xi|^{\gamma^+}} \mathrm{d}\xi \geq \frac{c_4^2}{n^{2+\gamma^+}} \int_n^{n+1} \mathrm{e}^{2\Phi(r_1,\xi)} \mathrm{d}\xi \geq \frac{c_4^2}{n^{2+\gamma^+}} \mathrm{e}^{2\Phi(r_1,n)} \end{aligned}$$

This shows that

$$\lim_{n\to+\infty} \|u_{2n}(r,\cdot) - u_2(r,\cdot)\| \geq c_4 \lim_{n\to+\infty} \frac{\mathrm{e}^{\Phi(r_1,n)}}{n^{1+\frac{\gamma^+}{2}}} = +\infty.$$

The proof is finished. □

4 Determination of the temperature distribution

In this section, a regularization scheme for ill-posed problem (2) is provided. In fact, the instability of solution caused by exponentially increasing in the term $\mathrm{e}^{|\xi|^{\frac{\gamma_1}{2}}}$, $\mathrm{e}^{|\xi|^{\frac{\gamma_2}{2}}}$ as $|\xi| \to \infty$. Therefore, to obtain the stability solution, the natural idea arising is to cutoff the high frequency term in solution. The selected cutoff point is required to satisfy the following criterion:

- It makes the regularized solution to maintain a good approximation to the exact one.
- It makes the regularized solution stable.

Specifically, the regularization solution have Fourier transform as follows:

$$\widehat{u^\delta_{2,\beta(\delta)}}(r,\xi) = \widehat{A}(r,\xi)\,\widehat{\varphi^\delta}(\xi)\,\chi_{\beta(\delta)}(\xi), \tag{14}$$

where $\beta(\delta)$ plays the role as the regularization parameter to be defined later, $\chi_{\beta(\delta)}$ denotes the characteristic function on the interval $[-\beta(\delta), \beta(\delta)]$, that is

$$\chi_{\beta(\delta)}(\xi) = \begin{cases} 1, & -\beta(\delta) \leq \xi \leq \beta(\delta), \\ 0, & |\xi| > \beta(\delta). \end{cases}$$

Then, we have regularized solution of problem (2):

$$u^\delta_{2,\beta(\delta)}(r,t) = \frac{1}{\sqrt{2\pi}} \int_{|\xi|\leq\beta(\delta)} \widehat{A}(r,\xi)\,\widehat{\varphi^\delta}(\xi)\,\mathrm{e}^{it\xi}\mathrm{d}\xi. \tag{15}$$

Theorem 4.1 (Stability of the regularized solution) *Let $u^\delta_{2,\beta(\delta)}$ and $u_{2,\beta(\delta)}$ be two regularized solution of problem (2) corresponding to the data φ^δ and φ, respectively, such that $\left\|\varphi^\delta - \varphi\right\| \leq \delta$. Then we have the following stable estimate:*

$$\left\|u^\delta_{2,\beta(\delta)}(r,\cdot) - u_{2,\beta(\delta)}(r,\cdot)\right\| \leq \left(c_1 \mathrm{e}^{\Psi(R,\overline{\theta}_{\max})} + c_2 \frac{\mathrm{e}^{\Phi(r,\beta(\delta))}}{(\beta(\delta))^{\frac{\gamma^-}{2}}} \right) \delta, \tag{16}$$

where $\overline{\theta}_{\max} = \max\left\{1, \left(\frac{\sqrt{\alpha_1}}{r_1\cos\left(\frac{\pi}{4}\gamma_1\right)}\right)^{\frac{2}{\gamma_1}}, \theta_{\max}\right\}$.

Proof By applying Parseval's identity, Lemmas 2.2 and 2.3, it follows the following sequential results:

$$\begin{aligned}
\left\|u^{\delta}_{2,\beta(\delta)}(r,\cdot) - u_{2,\beta(\delta)}(r,\cdot)\right\|^2 &= \left\|\widehat{u}^{\delta}_{2,\beta(\delta)}(r,\cdot) - \widehat{u}_{2,\beta(\delta)}(r,\cdot)\right\|^2 \\
&= \int_{|\xi|\le\overline{\theta}_{\max}} \left|\widehat{A}(r,\xi)\right|^2\left|\widehat{\varphi^{\delta}}(\xi) - \widehat{\varphi}(\xi)\right|^2 \mathrm{d}\xi \\
&\quad + \int_{\overline{\theta}_{\max}<|\xi|\le\beta(\delta)} \left|\widehat{A}(r,\xi)\right|^2\left|\widehat{\varphi^{\delta}}(\xi) - \widehat{\varphi}(\xi)\right|^2 \mathrm{d}\xi \\
&\le c_1^2 \int_{|\xi|\le\overline{\theta}_{\max}} \mathrm{e}^{2\Psi(r,\xi)}\left|\widehat{\varphi^{\delta}}(\xi) - \widehat{\varphi}(\xi)\right|^2 \mathrm{d}\xi \\
&\quad + c_2^2 \int_{\overline{\theta}_{\max}<|\xi|\le\beta(\delta)} \frac{\mathrm{e}^{2\Phi(r,\xi)}}{|\xi|^{\gamma^-}}\left|\widehat{\varphi^{\delta}}(\xi) - \widehat{\varphi}(\xi)\right|^2 \mathrm{d}\xi \\
&\le c_1^2 \mathrm{e}^{2\Psi\left(R,\overline{\theta}_{\max}\right)} \int_{\mathbb{R}} \left|\widehat{\varphi^{\delta}}(\xi) - \widehat{\varphi}(\xi)\right|^2 \mathrm{d}\xi \\
&\quad + c_2^2 \sup_{|\xi|\in\left(\overline{\theta}_{\max},\beta(\delta)\right]} \frac{\mathrm{e}^{2\Phi(r,\xi)}}{|\xi|^{\gamma^-}} \int_{\mathbb{R}} \left|\widehat{\varphi^{\delta}}(\xi) - \widehat{\varphi}(\xi)\right|^2 \mathrm{d}\xi \\
&\le c_1^2 \mathrm{e}^{2\Psi\left(R,\overline{\theta}_{\max}\right)}\delta^2 + c_2^2 \sup_{|\xi|\in\left(\overline{\theta}_{\max},\beta(\delta)\right]} \frac{\mathrm{e}^{2\Phi(r,\xi)}}{|\xi|^{\gamma^-}}\delta^2.
\end{aligned}$$

Let $z = |\xi|$ and $\overline{\Phi}(r,z) = \frac{r_1}{\sqrt{\alpha_1}}\cos\left(\frac{\pi}{4}\gamma_1\right)z^{\frac{\gamma_1}{2}} + \frac{r-r_1}{\sqrt{\alpha_2}}\cos\left(\frac{\pi}{4}\gamma_2\right)z^{\frac{\gamma_2}{2}}$. It follows that $\frac{\mathrm{e}^{2\Phi(r,\xi)}}{|\xi|^{\gamma^-}} = \frac{\mathrm{e}^{2\overline{\Phi}(r,z)}}{z^{\gamma^-}} =: f(z)$. Then we have

$$\begin{aligned}
f'(z) &= \frac{\mathrm{e}^{2\overline{\Phi}(r,z)}}{z^{\gamma^-+1}}\left(\gamma_1\frac{r_1}{\sqrt{\alpha_1}}\cos\left(\frac{\pi}{4}\gamma_1\right)z^{\frac{\gamma_1}{2}} + \gamma_2\frac{r-r_1}{\sqrt{\alpha_2}}\cos\left(\frac{\pi}{4}\gamma_2\right)z^{\frac{\gamma_2}{2}} - \gamma^-\right) \\
&\ge \frac{\mathrm{e}^{2\overline{\Phi}(r,z)}}{z^{\gamma^-+1}}\left(\gamma_1\frac{r_1}{\sqrt{\alpha_1}}\cos\left(\frac{\pi}{4}\gamma_1\right)z^{\frac{\gamma_1}{2}} - \gamma^-\right).
\end{aligned}$$

From $z = |\xi| > \overline{\theta}_{\max} \ge \left(\frac{\sqrt{\alpha_1}}{r_1\cos\left(\frac{\pi}{4}\gamma_1\right)}\right)^{\frac{2}{\gamma_1}}$ we deduce that $f'(z) \ge \frac{\mathrm{e}^{2\overline{\Phi}(r,z)}}{z^{\gamma^-+1}}\left(\gamma_1 - \gamma^-\right) \ge 0$. It follows that

$$\frac{\mathrm{e}^{2\Phi(r,\xi)}}{|\xi|^{\gamma^-}} \le f(\beta(\delta)) = \frac{\mathrm{e}^{2\Phi(r,\beta(\delta))}}{(\beta(\delta))^{\gamma^-}} \quad \text{for all } |\xi| \in \left(\overline{\theta}_{\max}, \beta(\delta)\right].$$

This means that

$$\left\|u^{\delta}_{2,\beta(\delta)}(r,\cdot) - u_{2,\beta(\delta)}(r,\cdot)\right\|^2 \le c_1^2\mathrm{e}^{2\Psi\left(R,\overline{\theta}_{\max}\right)}\delta^2 + c_2^2\frac{\mathrm{e}^{2\Phi(r,\beta(\delta))}}{(\beta(\delta))^{\gamma^-}}\delta^2.$$

Therefore

$$\left\|u^{\delta}_{2,\beta(\delta)}(r,\cdot) - u_{2,\beta(\delta)}(r,\cdot)\right\| \le \left(c_1\mathrm{e}^{\Psi\left(R,\overline{\theta}_{\max}\right)} + c_2\frac{\mathrm{e}^{\Phi(r,\beta(\delta))}}{(\beta(\delta))^{\frac{\gamma^-}{2}}}\right)\delta.$$

The proof is finished. □

Theorem 4.2 *Let u_2 be the solution of problem (2) and $u^{\delta}_{2,\beta(\delta)}$ be the regularized solution given by (15) and the measured data φ^{δ} satisfy (5). If the exact solution u_2 satisfies (6) for $p > \frac{1}{2}\left(\gamma^{+} - \gamma^{-}\right)$. The regularization parameter $\beta(\delta)$ be selected by*

$$\beta(\delta) = \begin{cases} \left(\frac{1}{\Theta(R)}\left(\ln\frac{\overline{E}_1}{\delta} - \left(\frac{2p+\gamma_1}{\gamma_2} - 1\right)\ln\left(\ln\frac{\overline{E}_1}{\delta}\right)\right)\right)^{\frac{2}{\gamma_2}}, & \text{if } \gamma_1 \le \gamma_2, \\ \left(\frac{1}{\Theta(R)}\ln\frac{\overline{E}_2}{\delta}\right)^{\frac{2}{\gamma_1}}, & \text{if } \gamma_1 > \gamma_2, \end{cases} \tag{17}$$

where

$$\Theta(r) = \frac{r_1}{\sqrt{\alpha_1}}\cos\left(\frac{\pi}{4}\gamma_1\right) + \frac{r - r_1}{\sqrt{\alpha_2}}\cos\left(\frac{\pi}{4}\gamma_2\right),$$
$$\overline{E}_1 = E + \delta\left(e^{e^{4\left(\frac{2p+\gamma_1}{\gamma_2}-1\right)}} + e^{2\Theta(R)\left(\overline{\theta}_{\max}\right)^{\frac{\gamma_2}{2}}}\right),$$
$$\overline{E}_2 = E + \delta e^{\Theta(R)\left(\overline{\theta}_{\max}\right)^{\frac{\gamma_1}{2}}}.$$

Then, for every $r \in (r_1, R]$, we obtain a convergence estimate:

$$d_1(r) \le \begin{cases} c_5\left(\delta + \overline{E}_1^{\frac{\Theta(r)}{\Theta(R)}}\delta^{1-\frac{\Theta(r)}{\Theta(R)}}\left(\ln\frac{\overline{E}_1}{\delta}\right)^{\left(1-\frac{2p+\gamma_1}{\gamma_2}\right)\frac{\Theta(r)}{\Theta(R)}}\right), & \text{if } \gamma_1 \le \gamma_2, \\ c_1 e^{\Psi\left(R,\overline{\theta}_{\max}\right)}\delta + c_2(\Theta(R))^{\frac{\gamma_2}{\gamma_1}}\overline{E}_2^{\frac{\Theta(r)}{\Theta(R)}}\delta^{1-\frac{\Theta(r)}{\Theta(R)}}\left(\ln\frac{\overline{E}_2}{\delta}\right)^{-\frac{\gamma_2}{\gamma_1}} & \\ \quad + \frac{c_2}{c_4}(\Theta(R))^{\frac{2p+\gamma_2}{\gamma_1}-1}E(\eta(\delta))^{\frac{\Theta(R)-\Theta(r)}{(\Theta(R))^{\frac{\gamma_2}{\gamma_1}}}}\left(\ln\frac{\overline{E}_2}{\delta}\right)^{1-\frac{2p+\gamma_2}{\gamma_1}}, & \text{if } \gamma_1 > \gamma_2, \end{cases} \tag{18}$$

where

$$d_1(r) = \left\| u^{\delta}_{2,\beta(\delta)}(r,\cdot) - u_2(r,\cdot)\right\|,$$
$$c_5 = \max\left\{c_1 e^{\Psi\left(R,\overline{\theta}_{\max}\right)}, c_2 + \frac{c_2}{c_4}(2\Theta(R))^{\left(\frac{2p+\gamma_1}{\gamma_2}-1\right)}\right\},$$
$$\eta(\delta) = e^{-\left(\ln\frac{\overline{E}_2}{\delta}\right)^{\frac{\gamma_2}{\gamma_1}}}.$$

Proof Using triangle inequality and Parseval's equality, we obtain that

$$d_1(r) \le \left\| u^{\delta}_{2,\beta(\delta)}(r,\cdot) - u_{2,\beta(\delta)}(r,\cdot)\right\| + \left\|\widehat{u}_{2,\beta(\delta)}(r,\cdot) - \widehat{u}_2(r,\cdot)\right\|. \tag{19}$$

(1) For $\left\| u^{\delta}_{2,\beta(\delta)}(r,\cdot) - u_{2,\beta(\delta)}(r,\cdot)\right\|$. In view of Theorem 4.1, one has

$$\left\| u^{\delta}_{2,\beta(\delta)}(r,\cdot) - u_{2,\beta(\delta)}(r,\cdot)\right\| \le c_1 e^{\Psi\left(R,\overline{\theta}_{\max}\right)}\delta + c_2\frac{e^{\Phi(r,\beta(\delta))}}{(\beta(\delta))^{\frac{\gamma^-}{2}}}\delta. \tag{20}$$

(2) For $\left\|\widehat{u}_{2,\beta(\delta)}(r,\cdot)-\widehat{u}_2(r,\cdot)\right\|$. Using Lemmas 2.3, 2.4 and (6), we obtain the following sequential results

$$\begin{aligned}
&\left\|\widehat{u}_{2,\beta(\delta)}(r,\cdot)-\widehat{u}_2(r,\cdot)\right\| \\
&=\left(\int_{|\xi|>\beta(\delta)}\frac{1}{\left(1+\xi^2\right)^p}\left|\frac{\widehat{A}(r,\xi)}{\widehat{A}(R,\xi)}\right|^2\left(1+\xi^2\right)^p\left|\widehat{A}(R,\xi)\widehat{\varphi}(\xi)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}} \\
&\le \sup_{|\xi|>\beta(\delta)}\frac{1}{|\xi|^p}\left|\frac{\widehat{A}(r,\xi)}{\widehat{A}(R,\xi)}\right|\left\|u_2(R,\cdot)\right\|_{H^p(\mathbb{R})} \qquad (21)\\
&\le \frac{c_2}{c_4}\sup_{|\xi|>\beta(\delta)}\frac{1}{|\xi|^{\frac{1}{2}(2p+\gamma^- -\gamma^+)}}\mathrm{e}^{(\Theta(r)-\Theta(R))|\xi|^{\frac{\gamma_2}{2}}}E \\
&\le \frac{c_2}{c_4(\beta(\delta))^{\frac{1}{2}(2p+\gamma^- -\gamma^+)}}\mathrm{e}^{(\Theta(r)-\Theta(R))(\beta(\delta))^{\frac{\gamma_2}{2}}}E.
\end{aligned}$$

Between γ_1 and γ_2 there are two cases.

In case 1: $\gamma_1\le\gamma_2$. Using (17), we obtain that

$$\left\|u^{\delta}_{2,\beta(\delta)}(r,\cdot)-u_{2,\beta(\delta)}(r,\cdot)\right\|\le c_1\mathrm{e}^{\Psi(R,\overline{\theta}_{\max})}\delta+c_2\overline{E}_1^{\frac{\Theta(r)}{\Theta(R)}}\delta^{1-\frac{\Theta(r)}{\Theta(R)}}\left(\ln\frac{\overline{E}_1}{\delta}\right)^{\left(1-\frac{2p+\gamma_1}{\gamma_2}\right)\frac{\Theta(r)}{\Theta(R)}}, \qquad (22)$$

$$\left\|\widehat{u}_{2,\beta(\delta)}(r,\cdot)-\widehat{u}_2(r,\cdot)\right\|\le\frac{c_2}{c_4(\beta(\delta))^{\frac{1}{2}(2p+\gamma_1-\gamma_2)}}\overline{E}_1^{\frac{\Theta(r)}{\Theta(R)}}\delta^{1-\frac{\Theta(r)}{\Theta(R)}}\left(\ln\frac{\overline{E}_1}{\delta}\right)^{\left(\frac{2p+\gamma_1}{\gamma_2}-1\right)\frac{\Theta(R)-\Theta(r)}{\Theta(R)}}. \qquad (23)$$

Now, we evaluate for the element $\beta(\delta)$. We claim that

$$\ln\frac{\overline{E}_1}{\delta}\ge 2\left(\frac{2p+\gamma_1}{\gamma_2}-1\right)\ln\left(\ln\frac{\overline{E}_1}{\delta}\right). \qquad (24)$$

Let $\alpha=4\left(\frac{2p+\gamma_1}{\gamma_2}-1\right)$ and $y=\ln\frac{\overline{E}_1}{\delta}>\mathrm{e}^{\alpha}$. Rather than solving (24), it is equivalent to solve the inequality $h(y)=y-\frac{\alpha}{2}\ln(y)\ge 0$ for $\alpha\ge 0$, $y>\mathrm{e}^{\alpha}$. In fact, from $h'(y)=1-\frac{\alpha}{2y}>1-\frac{\alpha}{2\mathrm{e}^{\alpha}}>0$, it follows that $h(y)>h(\mathrm{e}^{\alpha})=\mathrm{e}^{\alpha}-\frac{\alpha^2}{2}\ge 0$. Therefore, the estimate (24) holds. Therefore

$$\beta(\delta)\ge\left(\frac{1}{2\Theta(R)}\left(\ln\frac{\overline{E}_1}{\delta}\right)\right)^{\frac{2}{\gamma_2}}.$$

This shows that

$$\left\|\widehat{u}_{2,\beta(\delta)}(r,\cdot)-\widehat{u}_2(r,\cdot)\right\|\le\frac{c_2}{c_4}(2\Theta(R))^{\left(\frac{2p+\gamma_1}{\gamma_2}-1\right)}\overline{E}_1^{\frac{\Theta(r)}{\Theta(R)}}\delta^{1-\frac{\Theta(r)}{\Theta(R)}}\left(\ln\frac{\overline{E}_1}{\delta}\right)^{\left(1-\frac{2p+\gamma_1}{\gamma_2}\right)\frac{\Theta(r)}{\Theta(R)}} \qquad (25)$$

Gathering (19), (22) and (25), we obtain the conclusion (18).

In case 2: $\gamma_1 > \gamma_2$. Using (20), (21) and (17), we obtain that

$$\left\| u^{\delta}_{2,\beta(\delta)}(r,\cdot) - u_{2,\beta(\delta)}(r,\cdot) \right\| \le c_1 \mathrm{e}^{\Psi(R,\overline{\theta}_{\max})}\delta + c_2(\Theta(R))^{\frac{\gamma_2}{\gamma_1}} \overline{E}_2^{\frac{\Theta(r)}{\Theta(R)}} \delta^{1-\frac{\Theta(r)}{\Theta(R)}} \left(\ln \frac{\overline{E}_2}{\delta} \right)^{-\frac{\gamma_2}{\gamma_1}}, \tag{26}$$

$$\left\| \widehat{u}_{2,\beta(\delta)}(r,\cdot) - \widehat{u}_2(r,\cdot) \right\| \le \frac{c_2}{c_4} (\Theta(R))^{\frac{2p+\gamma_2}{\gamma_1}-1} E(\eta(\delta))^{\frac{\Theta(R)-\Theta(r)}{(\Theta(R))^{\frac{\gamma_2}{\gamma_1}}}} \left(\ln \frac{\overline{E}_2}{\delta} \right)^{1-\frac{2p+\gamma_2}{\gamma_1}}. \tag{27}$$

Combining (19), (26) and (27), we get the conclusion (18).
The proof is finished. □

Remark 4.1 If $\gamma_1 = \gamma_2 = 1$ and $\eta_1 = \eta_2 = 0$, then

$$\left\| u^{\delta}_{2,\beta(\delta)}(r,\cdot) - u_2(r,\cdot) \right\| \le c_5 \left(\delta + \overline{E}_1^{\frac{\Theta(r)}{\Theta(R)}} \delta^{1-\frac{\Theta(r)}{\Theta(R)}} \left(\ln \frac{\overline{E}_1}{\delta} \right)^{-\frac{2p\Theta(r)}{\Theta(R)}} \right) \quad \text{for all } p > 0.$$

This result is an extension for Theorem 3.1 in Xiong et al. (2016).

5 Determination of the heat flux distribution

Based on idea cut-off as Sect. 4, we have Fourier transform of regularized heat flux solution of problem (2) as follows:

$$\frac{\partial \widehat{u}^{\delta}_{2,\overline{\beta}(\delta)}(r,\xi)}{\partial r} = \widehat{X}(r,\xi)\,\widehat{\varphi^{\delta}}(\xi)\,\chi_{\overline{\beta}(\delta)}(\xi), \tag{28}$$

where $\overline{\beta}(\delta)$ plays the role as the regularization parameter to be defined later, $\chi_{\overline{\beta}(\delta)}$ denotes the characteristic function on the interval $\left[-\overline{\beta}(\delta), \overline{\beta}(\delta)\right]$. Then we have regularized heat flux solution of problem (2):

$$\frac{\partial u^{\delta}_{2,\overline{\beta}(\delta)}(r,t)}{\partial r} = \frac{1}{\sqrt{2\pi}} \int_{|\xi| \le \overline{\beta}(\delta)} \widehat{X}(r,\xi)\,\widehat{\varphi^{\delta}}(\xi)\,\mathrm{e}^{it\xi}\,\mathrm{d}\xi, \tag{29}$$

Theorem 5.1 (Stability of the regularized heat flux solution) *Let* $\frac{\partial u^{\delta}_{2,\overline{\beta}(\delta)}}{\partial r}$ *and* $\frac{\partial u_{2,\overline{\beta}(\delta)}}{\partial r}$ *be two regularized heat flux solution of problem* (2) *corresponding to the data* φ^{δ} *and* φ, *respectively, such that* $\left\|\varphi^{\delta} - \varphi\right\| \le \delta$. *Then we have the following stable estimate*:

$$\left\| \frac{\partial u^{\delta}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r} - \frac{\partial u_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r} \right\| \le \left(\overline{c}_1\left(\widehat{\theta}_{\max}\right) \mathrm{e}^{\Psi(R,\widehat{\theta}_{\max})} + c_3 \mathrm{e}^{\Theta(r)\left(\overline{\beta}(\delta)\right)^{\frac{\gamma^+}{2}}} \right) \delta, \tag{30}$$

where

$$\widehat{\theta}_{\max} = \begin{cases} \max\{1, \theta_{\max}\}, & \text{if } \gamma_1 \ge \gamma_2, \\ \max\left\{ 2^{\frac{2}{\gamma_2-\gamma_1}}, \left(\frac{2\sqrt{\alpha_1}}{r_1 \cos\left(\frac{\pi}{4}\gamma_1\right)} \right)^{\frac{2}{\gamma_1}}, \theta_{\max} \right\}, & \text{if } \gamma_1 < \gamma_2. \end{cases}$$

Proof Using Parseval's identity, Lemmas 2.2 and 2.3, we obtain that

$$\begin{aligned}\left\|\frac{\partial\widehat{u}^{\delta}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial\widehat{u}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}\right\|^2 &= \int_{|\xi|\le\widehat{\theta}_{\max}}\left|\widehat{X}(r,\xi)\right|^2\left|\widehat{\varphi^{\delta}}(\xi)-\widehat{\varphi}(\xi)\right|^2\mathrm{d}\xi\\ &+\int_{\widehat{\theta}_{\max}<|\xi|\le\overline{\beta}(\delta)}\left|\widehat{X}(r,\xi)\right|^2\left|\widehat{\varphi^{\delta}}(\xi)-\widehat{\varphi}(\xi)\right|^2\mathrm{d}\xi\\ &\le\overline{c}_1^2\left(\widehat{\theta}_{\max}\right)\int_{|\xi|\le\widehat{\theta}_{\max}}\mathrm{e}^{2\Psi(r,\xi)}\left|\widehat{\varphi^{\delta}}(\xi)-\widehat{\varphi}(\xi)\right|^2\mathrm{d}\xi\\ &+c_3^2\int_{\widehat{\theta}_{\max}<|\xi|\le\overline{\beta}(\delta)}|\xi|^{\gamma^*}\mathrm{e}^{2\Phi(r,\xi)}\left|\widehat{\varphi^{\delta}}(\xi)-\widehat{\varphi}(\xi)\right|^2\mathrm{d}\xi\\ &\le\overline{c}_1^2\left(\widehat{\theta}_{\max}\right)\mathrm{e}^{2\Psi\left(R,\widehat{\theta}_{\max}\right)}\delta^2+c_3^2\sup_{|\xi|\in\left(\widehat{\theta}_{\max},\overline{\beta}(\delta)\right]}|\xi|^{\gamma^*}\mathrm{e}^{2\Phi(r,\xi)}\delta^2.\end{aligned}$$

This implies that

$$\left\|\frac{\partial\widehat{u}^{\delta}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial\widehat{u}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}\right\|\le\overline{c}_1\left(\widehat{\theta}_{\max}\right)\mathrm{e}^{\Psi\left(R,\widehat{\theta}_{\max}\right)}\delta+c_3\sup_{|\xi|\in\left(\widehat{\theta}_{\max},\overline{\beta}(\delta)\right]}|\xi|^{\frac{\gamma^*}{2}}\mathrm{e}^{\Phi(r,\xi)}\delta.$$

Between γ_1 and γ_2 there are two cases:
In case 1: $\gamma_1\ge\gamma_2$. It is easy to see that

$$\left\|\frac{\partial\widehat{u}^{\delta}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial\widehat{u}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}\right\|\le\left(\overline{c}_1\left(\widehat{\theta}_{\max}\right)\mathrm{e}^{\Psi\left(R,\widehat{\theta}_{\max}\right)}+c_3\mathrm{e}^{\Theta(r)\left(\overline{\beta}(\delta)\right)^{\frac{\gamma^+}{2}}}\right)\delta.$$

In case 2: $\gamma_1<\gamma_2$. It is known that

$$\left\|\frac{\partial\widehat{u}^{\delta}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial\widehat{u}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}\right\|\le\overline{c}_1\left(\widehat{\theta}_{\max}\right)\mathrm{e}^{\Psi\left(R,\widehat{\theta}_{\max}\right)}\delta+c_3\sup_{|\xi|\in\left(\widehat{\theta}_{\max},\overline{\beta}(\delta)\right]}|\xi|^{\frac{\gamma_2-\gamma_1}{2}}\mathrm{e}^{\Phi(r,\xi)}\delta.$$

Since $|\xi|>\widehat{\theta}_{\max}\ge\max\left\{2^{\frac{2}{\gamma_2-\gamma_1}},\left(\frac{2\sqrt{\alpha_1}}{r_1\cos\left(\frac{\pi}{4}\gamma_1\right)}\right)^{\frac{2}{\gamma_1}}\right\}$, it follows that

$$\begin{aligned}|\xi|^{\frac{\gamma_2-\gamma_1}{2}}\mathrm{e}^{\Phi(r,\xi)} &= |\xi|^{\frac{\gamma_2-\gamma_1}{2}}\mathrm{e}^{\frac{r_1}{\sqrt{\alpha_1}}\cos\left(\frac{\pi}{4}\gamma_1\right)|\xi|^{\frac{\gamma_1}{2}}+\frac{r-r_1}{\sqrt{\alpha_2}}\cos\left(\frac{\pi}{4}\gamma_2\right)|\xi|^{\frac{\gamma_2}{2}}}\\ &\le\mathrm{e}^{|\xi|^{\frac{\gamma_2-\gamma_1}{2}}+\frac{r_1}{\sqrt{\alpha_1}}\cos\left(\frac{\pi}{4}\gamma_1\right)|\xi|^{\frac{\gamma_1}{2}}+\frac{r-r_1}{\sqrt{\alpha_2}}\cos\left(\frac{\pi}{4}\gamma_2\right)|\xi|^{\frac{\gamma_2}{2}}}\\ &\le\mathrm{e}^{\Theta(r)\left(\overline{\beta}(\delta)\right)^{\frac{\gamma_2}{2}}}=\mathrm{e}^{\Theta(r)\left(\overline{\beta}(\delta)\right)^{\frac{\gamma^+}{2}}}.\end{aligned}$$

This shows that

$$\left\|\frac{\partial\widehat{u}^{\delta}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial\widehat{u}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}\right\|\le\left(\overline{c}_1\left(\widehat{\theta}_{\max}\right)\mathrm{e}^{\Psi\left(R,\widehat{\theta}_{\max}\right)}+c_3\mathrm{e}^{\Theta(r)\left(\overline{\beta}(\delta)\right)^{\frac{\gamma^+}{2}}}\right)\delta.$$

The proof is finished. □

Theorem 5.2 *Let* $\frac{\partial u_2}{\partial r}$ *be the heat flux solution of problem* (2) *and* $\frac{\partial u^{\delta}_{2,\overline{\beta}(\delta)}}{\partial r}$ *be the regularized solution given by* (29) *and the measured data* φ^{δ} *satisfy* (5). *If the exact solution* u_2 *satisfies*

(6) *for* $p > \frac{1}{2}\left(\gamma^+ + \gamma^*\right)$. *The regularization parameter* $\overline{\beta}(\delta)$ *be selected by*

$$\overline{\beta}(\delta) = \begin{cases} \left(\frac{1}{\Theta(R)}\left(\ln\frac{\overline{E}_3}{\delta} - \left(\frac{2p+\gamma_1}{\gamma_2} - 2\right)\ln\left(\ln\frac{\overline{E}_3}{\delta}\right)\right)\right)^{\frac{2}{\gamma_2}}, & \text{if } \gamma_1 \le \gamma_2, \\ \left(\frac{1}{\Theta(R)}\left(\ln\frac{\overline{E}_4}{\delta} - \left(\frac{2p}{\gamma_1} - 1\right)\ln\left(\ln\frac{\overline{E}_4}{\delta}\right)\right)\right)^{\frac{2}{\gamma_1}}, & \text{if } \gamma_1 > \gamma_2, \end{cases} \tag{31}$$

in which

$$\overline{E}_3 = E + \delta\left(\mathrm{e}^{\mathrm{e}^{4\left(\frac{2p+\gamma_1}{\gamma_2}-2\right)}} + \mathrm{e}^{2\Theta(R)\left(\widehat{\theta}_{\max}\right)^{\frac{\gamma_2}{2}}}\right),$$

$$\overline{E}_4 = E + \delta\left(\mathrm{e}^{\mathrm{e}^{4\left(\frac{2p}{\gamma_1}-1\right)}} + \mathrm{e}^{2\Theta(R)\left(\widehat{\theta}_{\max}\right)^{\frac{\gamma_1}{2}}}\right).$$

Then for every $r \in (r_1, R]$, *we obtain a convergence estimate*:

$$d_2(r) \le \begin{cases} c_6\left(\delta + \overline{E}_3^{\frac{\Theta(r)}{\Theta(R)}}\delta^{1-\frac{\Theta(r)}{\Theta(R)}}\left(\ln\frac{\overline{E}_3}{\delta}\right)^{\left(2-\frac{2p+\gamma_1}{\gamma_2}\right)\frac{\Theta(r)}{\Theta(R)}}\right), & \text{if } \gamma_1 \le \gamma_2, \\ \overline{c}_1\left(\widehat{\theta}_{\max}\right)\mathrm{e}^{\Psi(R,\widehat{\theta}_{\max})}\delta + c_3\overline{E}_4^{\frac{\Theta(r)}{\Theta(R)}}\delta^{1-\frac{\Theta(r)}{\Theta(R)}}\left(\ln\frac{\overline{E}_4}{\delta}\right)^{\left(1-\frac{2p}{\gamma_1}\right)\frac{\Theta(r)}{\Theta(R)}} & \\ +\frac{c_3}{c_4}(2\Theta(R))^{\frac{2p}{\gamma_1}-1}E(\overline{\eta}(\delta))^{\frac{\Theta(R)-\Theta(r)}{(2\Theta(R))^{\frac{\gamma_2}{\gamma_1}}}}\left(\ln\frac{\overline{E}_4}{\delta}\right)^{1-\frac{2p}{\gamma_1}}, & \text{if } \gamma_1 > \gamma_2, \end{cases} \tag{32}$$

where

$$d_2(r) = \left\|\frac{\partial u^{\delta}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r} - \frac{\partial u_2(r,\cdot)}{\partial r}\right\|,$$

$$c_6 = \max\left\{\overline{c}_1\left(\widehat{\theta}_{\max}\right)\mathrm{e}^{\Psi(R,\widehat{\theta}_{\max})}, c_3 + \frac{c_3}{c_4}(2\Theta(R))^{\left(\frac{2p+\gamma_1}{\gamma_2}-2\right)}\right\},$$

$$\overline{\eta}(\delta) = \mathrm{e}^{-\left(\ln\frac{E_4}{\delta}\right)^{\frac{\gamma_2}{\gamma_1}}}.$$

Proof By triangle inequality and Parseval's equality, we obtain that

$$d_2(r) \le \left\|\frac{\partial u^{\delta}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r} - \frac{\partial u_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}\right\| + \left\|\frac{\partial \widehat{u}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r} - \frac{\partial \widehat{u}_2(r,\cdot)}{\partial r}\right\|. \tag{33}$$

(1) For $\left\|\frac{\partial u^{\delta}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r} - \frac{\partial u_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}\right\|$. In view of Theorem 5.1, one has

$$\left\|\frac{\partial u^{\delta}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r} - \frac{\partial u_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}\right\| \le \overline{c}_1\left(\widehat{\theta}_{\max}\right)\mathrm{e}^{\Psi(R,\widehat{\theta}_{\max})}\delta + c_3\mathrm{e}^{\Theta(r)\left(\overline{\beta}(\delta)\right)^{\frac{\gamma^+}{2}}}\delta. \tag{34}$$

(2) For $\left\|\frac{\partial\widehat{u}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial\widehat{u}_2(r,\cdot)}{\partial r}\right\|$. Using Lemmas 2.3, 2.4 and (6), we obtain that

$$\begin{aligned}
&\left\|\frac{\partial\widehat{u}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial\widehat{u}_2(r,\cdot)}{\partial r}\right\|\\
&=\left(\int_{|\xi|>\overline{\beta}(\delta)}\frac{1}{\left(1+\xi^2\right)^p}\left|\frac{\widehat{X}(r,\xi)}{\widehat{A}(R,\xi)}\right|^2\left(1+\xi^2\right)^p\left|\widehat{A}(R,\xi)\widehat{\varphi}(\xi)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}\\
&\le\sup_{|\xi|>\overline{\beta}(\delta)}\frac{1}{|\xi|^p}\left|\frac{\widehat{X}(r,\xi)}{\widehat{A}(R,\xi)}\right|\|u_2(R,\cdot)\|_{H^p(\mathbb{R})} \qquad (35)\\
&\le\frac{c_3}{c_4}\sup_{|\xi|>\overline{\beta}(\delta)}\frac{1}{|\xi|^{\frac{1}{2}(2p-\gamma^+-\gamma^*)}}\mathrm{e}^{(\Theta(r)-\Theta(R))|\xi|^{\frac{\gamma_2}{2}}}E\\
&\le\frac{c_3}{c_4\left(\overline{\beta}(\delta)\right)^{\frac{1}{2}(2p-\gamma^+-\gamma^*)}}\mathrm{e}^{(\Theta(r)-\Theta(R))\left(\overline{\beta}(\delta)\right)^{\frac{\gamma_2}{2}}}E.
\end{aligned}$$

Between γ_1 and γ_2 there are two cases:
In case 1: $\gamma_1\le\gamma_2$. Using (31), we obtain that

$$\left\|\frac{\partial u^\delta_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial u_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}\right\|\le\overline{c}_1\left(\widehat{\theta}_{\max}\right)\mathrm{e}^{\Psi(R,\widehat{\theta}_{\max})}\delta+c_3\overline{E}_3^{\frac{\Theta(r)}{\Theta(R)}}\delta^{1-\frac{\Theta(r)}{\Theta(R)}}\left(\ln\frac{\overline{E}_3}{\delta}\right)^{\left(2-\frac{2p+\gamma_1}{\gamma_2}\right)\frac{\Theta(r)}{\Theta(R)}}, \qquad (36)$$

$$\left\|\frac{\partial\widehat{u}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial\widehat{u}_2(r,\cdot)}{\partial r}\right\|\le\frac{c_3}{c_4\left(\overline{\beta}(\delta)\right)^{\frac{1}{2}(2p+\gamma_1-2\gamma_2)}}\overline{E}_3^{\frac{\Theta(r)}{\Theta(R)}}\delta^{1-\frac{\Theta(r)}{\Theta(R)}}\left(\ln\frac{\overline{E}_3}{\delta}\right)^{\left(\frac{2p+\gamma_1}{\gamma_2}-2\right)\frac{\Theta(R)-\Theta(r)}{\Theta(R)}}. \qquad (37)$$

From $\overline{E}_3>\delta\mathrm{e}^{\mathrm{e}^{4\left(\frac{2p+\gamma_1}{\gamma_2}-2\right)}}$ it follows that $\overline{\beta}(\delta)\ge\left(\frac{1}{2\Theta(R)}\left(\ln\frac{\overline{E}_3}{\delta}\right)\right)^{\frac{2}{\gamma_2}}$. This shows that

$$\left\|\frac{\partial\widehat{u}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial\widehat{u}_2(r,\cdot)}{\partial r}\right\|\le\frac{c_3}{c_4}(2\Theta(R))^{\left(\frac{2p+\gamma_1}{\gamma_2}-2\right)}\overline{E}_3^{\frac{\Theta(r)}{\Theta(R)}}\delta^{1-\frac{\Theta(r)}{\Theta(R)}}\left(\ln\frac{\overline{E}_3}{\delta}\right)^{\left(2-\frac{2p+\gamma_1}{\gamma_2}\right)\frac{\Theta(r)}{\Theta(R)}}. \qquad (38)$$

Gathering (33), (36) and (38), we conclude the conclusion (32).
In case 2: $\gamma_1>\gamma_2$. In view of (34) and (35), one has

$$\left\|\frac{\partial u^\delta_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial u_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}\right\|\le\overline{c}_1\left(\widehat{\theta}_{\max}\right)\mathrm{e}^{\Psi(R,\widehat{\theta}_{\max})}\delta+c_3\mathrm{e}^{\Theta(r)\left(\overline{\beta}(\delta)\right)^{\frac{\gamma_1}{2}}}\delta,$$

$$\left\|\frac{\partial\widehat{u}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial\widehat{u}_2(r,\cdot)}{\partial r}\right\|\le\frac{c_3}{c_4\left(\overline{\beta}(\delta)\right)^{\frac{1}{2}(2p-\gamma_1)}}\mathrm{e}^{(\Theta(r)-\Theta(R))\left(\overline{\beta}(\delta)\right)^{\frac{\gamma_2}{2}}}E.$$

Since $\overline{E}_4>\delta\mathrm{e}^{\mathrm{e}^{4\left(\frac{2p}{\gamma_1}-1\right)}}$ it follows that $\overline{\beta}(\delta)\ge\left(\frac{1}{2\Theta(R)}\left(\ln\frac{\overline{E}_4}{\delta}\right)\right)^{\frac{2}{\gamma_1}}$. By using (31), we obtain that

$$\left\|\frac{\partial u^\delta_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial u_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}\right\|\le\overline{c}_1\left(\widehat{\theta}_{\max}\right)\mathrm{e}^{\Psi(R,\widehat{\theta}_{\max})}\delta+c_3\overline{E}_4^{\frac{\Theta(r)}{\Theta(R)}}\delta^{1-\frac{\Theta(r)}{\Theta(R)}}\left(\ln\frac{\overline{E}_4}{\delta}\right)^{\left(1-\frac{2p}{\gamma_1}\right)\frac{\Theta(r)}{\Theta(R)}}, \qquad (39)$$

$$\left\|\frac{\partial\widehat{u}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r}-\frac{\partial\widehat{u}_2(r,\cdot)}{\partial r}\right\|\le\frac{c_3}{c_4}(2\Theta(R))^{\frac{2p}{\gamma_1}-1}E(\overline{\eta}(\delta))^{\frac{\Theta(R)-\Theta(r)}{(2\Theta(R))^{\frac{\gamma_2}{\gamma_1}}}}\left(\ln\frac{\overline{E}_4}{\delta}\right)^{1-\frac{2p}{\gamma_1}}. \qquad (40)$$

Combining (33), (39) and (40), we get the conclusion (32).
The proof is finished. □

Remark 5.1 If $\gamma_1 = \gamma_2 = 1$ and $\eta_1 = \eta_2 = 0$, then

$$\left\| \frac{\partial u^{\delta}_{2,\overline{\beta}(\delta)}(r,\cdot)}{\partial r} - \frac{\partial u_2(r,\cdot)}{\partial r} \right\| \le c_6 \left(\delta + \overline{E}_3^{\frac{\Theta(r)}{\Theta(R)}} \delta^{1-\frac{\Theta(r)}{\Theta(R)}} \left(\ln \frac{\overline{E}_3}{\delta} \right)^{(1-2p)\frac{\Theta(r)}{\Theta(R)}} \right) \quad \text{for all } p > \frac{1}{2}.$$

This result is an extension for Theorem 3.2 in Xiong et al. (2016).

6 Concluding remark

In the current paper, I study an inverse boundary problem for the time-fractional diffusion equation in two layers spherical domain with a nonlinear boundary condition of Robin-type. I first prove the ill-posedness of the problem. Then, I proposed a truncation-type regularized solutions for both the temperature and the heat flux. This method yields a Hölder-type error with respect to the L^2-norm.

Acknowledgements I am grateful to the anonymous reviewers for their valuable suggestions to make my manuscript more completely.

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